Les Limites

$$\lim_{x \to a} f(x) = \ell.$$

1- Règles usuelles : "Addition"

- $+\infty + \infty = +\infty$
- $-\infty - \infty = -\infty$
- $+\infty + a = +\infty$
- $+\infty - a = +\infty$
- $-\infty + a = -\infty$
- $-\infty - a = -\infty$
- $-\infty + \infty = F.I$
- $+\infty - \infty = F.I$

2- Règles : multiplication

- $+\infty \times +\infty = +\infty$
- $+\infty \times -\infty = -\infty$
- $+\infty \times (-a) = -\infty$
- $+\infty \times (+a) = +\infty$
- $-\infty \times -\infty = +\infty$
- $-\infty \times (-a) = +\infty$
- $-\infty \times (+a) = -\infty$
- $\pm\infty \times 0 = F.I$

3- Règles : puissance

- $(+\infty)^2 = +\infty$
- $(+\infty)^n = +\infty \ (n \in \mathbb{Z})$
- $(-\infty)^{\mathbb{Z}} = +\infty$
- $(-\infty)^n$ $\begin{cases} n : \text{pair} \quad (-\infty)^n = +\infty \\ n : \text{impair} \quad (-\infty)^n = -\infty \end{cases}$

N° 1 Règle :

$$\lim_{x \to \pm\infty} \text{polynôme} = \lim_{x \to \infty} \text{plus grand puissance}$$

Exemple :

- $\lim_{x \to +\infty} 5x^3 + 4x^2 + 3x + 1 = \lim_{x \to +\infty} 5x^3 = +\infty$
- $\lim_{x \to -\infty} 1 + 4x^2 + 7x = \lim_{x \to -\infty} 4x^2 = 4(-\infty)^2 = +\infty$
- $\lim_{x \to 1} 3x^2 + 4x + 1 = \lim_{x \to 1} 3x^2 = 3$ (No Error)

N° 2 Règle.

$\lim_{x \to \infty} \sqrt{\text{polynôme}}$ => Et ①

$\lim_{x \to \infty} \text{polynôme}$

$= \lim_{x \to \infty} \text{plus grand puissance}$

Et ② :

$\lim_{x \to \infty} \sqrt{\text{poly}} =$

Exemple :

$$\lim_{x \to +\infty} \sqrt{2x^2+3x^3+2x+1}$$

. on calcule : $\lim_{x \to +\infty} 2x^2+3x^2+2x+1$

$$= \lim_{x \to +\infty} 3x^2 = +\infty$$

Donc : $\lim_{x \to +\infty} \sqrt{2x^2+3x^3+2x+1} = \sqrt{+\infty} = +\infty$

. $\lim_{x \to +\infty} \sqrt{\dfrac{14x^2+7x+2}{7x^2+4x+1}} = \lim_{x \to +\infty}$??

. on calcule : $\lim_{x \to +\infty} \dfrac{14x^2+7x+2}{7x^2+4x+1} = \lim_{x \to +\infty} \dfrac{14\cancel{x^2}}{7\cancel{x^2}}$

$$= \boxed{2}$$

Donc : $\lim_{x \to +\infty} \sqrt{\dfrac{14x^2+8x+2}{7x^2+4x+1}} = \sqrt{2}$.

4- Règles : Division

- $\frac{+\infty}{K} = +\infty$ si $K > 0$
- $\frac{+\infty}{K} = -\infty$ si $K < 0$
- $\frac{-\infty}{K} = +\infty$ si $K < 0$
- $\frac{-\infty}{K} = -\infty$ si $K > 0$
- $\frac{\pm\infty}{\pm\infty} = F.I$
- $\frac{K}{\pm\infty} = 0$

Note : $\frac{K}{\pm\infty} = 0$ Toujours.

- 4 formes indéterminés :

1. $\frac{0}{0}$; 2. $\frac{\infty}{\infty}$; 3. $0 \times \infty$; 4. $+\infty - \infty$; $-\infty - \infty$

a) forme « $\frac{0}{0}$ » :

$F.I \left(\frac{0}{0}\right) \xrightarrow{\text{factorisation}}$ forme déterminés (F.D)

① $\Delta = b^2 - 4 \times a \times c$

- $\Delta > 0$: $a.(x - x_1).(x - x_2)$
- $\Delta = 0$: $a.(x - x_0)^2$
- $\Delta < 0$: $\emptyset$

②

$\sqrt{a} + \sqrt{b} \longrightarrow \sqrt{a} - \sqrt{b}$

$\sqrt{a} + b \longrightarrow \sqrt{a} - b$

$\sqrt{a} \longrightarrow \sqrt{a}$

$b\sqrt{a} \longrightarrow \sqrt{a}$

③ division euclidienne

$$\begin{array}{c|c} A & B \\ \hline & C \\ = D & \end{array}$$

$A = B \times C + D$

Exemple : factorisation $\Delta = b^2 - 4ac$

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = \frac{0}{0}$$

on a : $x^2 + x - 2 = 0$

$$\begin{cases} a = 1 \\ b = 1 \\ c = -2 \end{cases}$$

$\Delta = b^2 - 4 \times a \times c$

$\Delta = 1^2 - 4 \times 1 \times (-2)$

$\Delta = 9 > 0$

$$x_1 = \frac{-b - \sqrt{\Delta}}{2 \times a} = \frac{-1 - \sqrt{9}}{2 \times (1)} = -2$$

$$x_2 = \frac{-b + \sqrt{\Delta}}{2 \times a} = \frac{-1 + \sqrt{9}}{2 \times 1} = 1$$

Donc : $x^2 + x - 2 = 1 \times (x - (-2))(x - 1)$

$= (x + 2)(x - 1)$

Alors. $$\lim_{x \to 1} \frac{(x+2)(x-1)}{x - 1} = \lim_{x \to 1} x + 2$$

$$= \boxed{3}$$

Exemple : factorisation

$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x \to 1} \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \to 1} \frac{(\sqrt{x})^2 - 1^2}{(x - 1)(\sqrt{x} + 1)}$$

$$= \lim_{x \to 1} \frac{(x - 1)}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \to 1} \frac{1}{\sqrt{x} + 1}$$

$$= \boxed{1/2}$$

Exa pk, factorisation · division euclidienne

$$\lim_{x \to 2} \frac{4x^3 - 5x^2 - 4x - 4}{x - 2} = \text{"}\frac{0}{0}\text{"}$$

$$\begin{array}{rrrr|l}
4x^3 & -5x^2 & -4x & -4 & x-2 \\
-4x^3 & +8x^2 & & & 4x^2+3x+2 \\
\hline
=0 & +3x^2 & -4x & -4 & \\
& -3x^2 & +6x & & \\
\hline
& =0 & +2x & -4 & \\
& & -2x & +4 & \\
\hline
& & 0 & 0 &
\end{array}$$

$$4x^3 - 5x^2 - 4x - 4 = (x-2)(4x^2+3x+2) + 0$$

Donc : $4x^3 - 5x^2 - 4x - 4 = (x-2)(4x^2+3x+2)$

Alors, $\lim_{x \to 2} \frac{(x-2)(4x^2+3x+2)}{x-2} = \lim_{x \to 2} 4x^2+3x+2$

$$= 4 \times (2)^2 + 3 \times 2 + 2 = \boxed{24}$$

Note : $\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{0}{0}$

problème $(x-a)$

$x \to 1$: $x - 1$

$x \to -2$: $x - (-2) = x + 2$

$x \to 0$: $x - 0 = x$

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b) forme $\frac{\pm\infty}{\pm\infty}$.

Règel ① . $\lim_{x \to \infty} \frac{P(x)}{Q(x)} = \lim_{x \to \infty} \frac{\text{plus grand puissance}}{\text{plus grand puissance}}$

Exemple

. $\lim_{x \to +\infty} \frac{4x^3 + 2x^2 + 5x - 1}{7x^2 + 8x + 1} = \lim_{x \to +\infty} \frac{4x^3}{7x^2} = \lim_{x \to +\infty} \frac{4x}{7}$

$= \boxed{+\infty}$

. $\lim_{x \to -\infty} \frac{8x^4 + 2x^3 + 20}{7x^3 + 2x^4 + 1} = \lim_{x \to -\infty} \frac{8x^4}{2x^4} = \frac{8}{2} = \boxed{4}$

. $\lim_{x \to +\infty} \frac{2x^3 + 4x^2 + 1}{4x^4 + 5x^2 + 10} = \lim_{x \to +\infty} \frac{2x^3}{4x^4} = \lim_{+\infty} \frac{2}{4x}$

$= \frac{2}{4 \times (+\infty)} = \frac{2}{+\infty} = 0$

Règle ② . $x \to +\infty$: $\sqrt{x^2} = x$

$x \to -\infty$: $\sqrt{x^2} = -x$

Exemple : $\lim_{x \to +\infty} \frac{\sqrt{4x^2 + 2x + 1}}{x^2 + 4x + 1} = \frac{+\infty}{+\infty}$ (F.I)

$= \lim_{x \to +\infty} \frac{\sqrt{x^2\left(4 + \frac{2}{x} + \frac{1}{x^2}\right)}}{x^2\left(1 + \frac{4}{x} + \frac{1}{x^2}\right)} = \lim_{x \to +\infty} \frac{\sqrt{x^2} \cdot \sqrt{4 + \frac{2}{x} + \frac{1}{x^2}}}{x^2 \cdot \left(1 + \frac{4}{x} + \frac{1}{x^2}\right)}$

Note : $x \to +\infty$: $\sqrt{x^2} = x$

$$= \lim_{x \to +\infty} \frac{x \cdot \sqrt{4 + \frac{2x}{x} + \frac{1}{x^2}}}{x^2\left(1 + \frac{4}{x} + \frac{1}{x^2}\right)} = \lim_{x \to +\infty} \frac{\sqrt{4 + \cancel{\frac{2}{x}} + \cancel{\frac{1}{x^2}}}}{x\left(1 + \cancel{\frac{4}{x}} + \cancel{\frac{1}{x^2}}\right)}$$

$$= \frac{\sqrt{4}}{+\infty \times 1} = 0$$

On $\lim\limits_{x \to +\infty} \frac{4}{x} = 0$ et $\lim\limits_{+\infty} \frac{1}{x^2} = 0$

Exemple ③ $\lim\limits_{x \to -\infty} \frac{3x^3 + 5x^2 + 4x + 1}{\sqrt{2x^2 + x + 1}} = \lim\limits_{x \to -\infty} \frac{x^2\left(3x + 5 + \frac{4}{x} + \frac{1}{x^2}\right)}{\sqrt{x^2\left(2 + \frac{1}{x} + \frac{1}{x^2}\right)}}$

$$= \lim_{x \to -\infty} \frac{x^2 \cdot \left(3x + 5 + \frac{4}{x} + \frac{1}{x^2}\right)}{\sqrt{x^2} \cdot \sqrt{2 + \frac{1}{x} + \frac{1}{x^2}}}$$

Note : $x \to -\infty$: $\sqrt{x^2} = -x$

$$= \lim_{x \to -\infty} \frac{x^2 \cdot \left(3x + 5 + \frac{4}{x} + \frac{1}{x^2}\right)}{-x \cdot \sqrt{2 + \frac{1}{x} + \frac{1}{x^2}}}$$

$$= \lim_{x \to -\infty} \frac{x \cdot \left(3x + 5 + \cancel{\frac{4}{x}} + \cancel{\frac{1}{x^2}}\right)}{-\sqrt{2 + \cancel{\frac{1}{x}} + \cancel{\frac{1}{x^2}}}}$$

$$= \frac{-\infty \times (3 \times (-\infty))}{-\sqrt{2}} = +\infty$$

• Limite à gauche et à Droite.

1/ Note : gauche de point a : a^{+}
Droite de point a : a^{+}

2/ Note : $\frac{K}{0^{+} \text{ou} -} = \pm\infty$; $\frac{K}{0^{+}} = +\infty$ Si $K > 0$
$\frac{K}{0^{+}} = -\infty$ Si $K < 0$

<u>Exemple</u> :

• $\lim\limits_{\substack{x \to 1 \\ x > 1}} \frac{2x+5}{x-1} = \frac{7}{0^{+}} = +\infty$

. $x - 1 = 0$
$x = \underline{1}$

x	$-\infty$	1^{-} 1 1^{+}	$+\infty$
$x-1$	$-$	0	$+$

• $\lim\limits_{\substack{x \to 2 \\ x < 2}} \frac{7x}{x^2-4} = \frac{14}{0}$

$x^2 - 4 = 0 \Leftrightarrow (x-2)(x+2) = 0$

$$x - 2 = 0 \quad \text{ou} \quad x + 2 = 0$$

$$x = 2 \quad \text{ou} \quad x = -2$$

x	$-\infty$		-2		2^- 2		$+\infty$
x^2-4		$+$	0	$-$	0	$+$	

Donc, $\lim\limits_{\substack{x \to 2 \\ x<2}} \frac{7x}{x^2-4} = \frac{14}{0^-} = -\infty$

• $\lim\limits_{\substack{x \to 0 \\ x<0}} \frac{x^3-6}{x^2-2x} = \frac{-6}{0}$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$x = 0 \quad \text{ou} \quad x - 2 = 0$$

$$x = 2$$

x	$-\infty$		0^- 0		2		$+\infty$
x^2-2x		$+$	0	$-$	0	$+$	

Donc: $\lim\limits_{\substack{x \to 0 \\ x<0}} \frac{x^3-6}{x^2-2x} = \frac{-6}{0^+} = -\infty$

<u>Note</u>. $\sqrt{0} = 0^+$; $0^2 = 0^+$; $|0| = 0^+$

<u>Toujours</u>.

Exemple :

- $\lim\limits_{\substack{x \to 1 \\ x>1}} \dfrac{-4x+2}{(x-1)^2} = \dfrac{-2}{0^2} = \dfrac{-2}{0^+} = -\infty$

- $\lim\limits_{\substack{x \to 2 \\ x<2}} \dfrac{2x}{\sqrt{x-2}} = \dfrac{4}{\sqrt{0}} = \dfrac{4}{0^+} = +\infty$

- $\lim\limits_{\substack{x \to 0 \\ x>0}} \dfrac{5x+1}{|x|} = \dfrac{1}{|0|} = \dfrac{1}{0^+} = +\infty$

- Régel : $\sin(\ )$; $\cos(\ )$, $\tan(\ )$

Régels N° 1 :

$$\begin{cases} \lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1 \\ \lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1 \\ \lim\limits_{x \to 0} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2} \end{cases}$$

Régel N° 2

$$\begin{cases} \lim\limits_{x \to 0} \dfrac{\sin f}{f} = 1 \\ \lim\limits_{x \to 0} \dfrac{\tan f}{f} = 1 \\ \lim\limits_{x \to 0} \dfrac{1-\cos f}{f^2} = 1/2 \end{cases}$$

Exemple :

- $\lim\limits_{x \to 0} \dfrac{\sin 4x}{x} = \lim\limits_{x \to 0} 4 \underbrace{\dfrac{\sin 4x}{4x}}_{\to 1} = 4 \times 1 = 4$

- $\lim\limits_{x \to 0} \dfrac{\sin 2x}{4x} = \lim\limits_{x \to 0} \underbrace{\dfrac{\sin 2x}{2 \times 2x}}_{\to 1} = \dfrac{1}{2}$

- $\lim\limits_{x \to 0} \dfrac{\sin 8x \times \tan 2x}{16x^2} = \lim\limits_{x \to 0} \dfrac{\sin 8x}{8x} \times \dfrac{\tan 2x}{2x}$
  $= 1 \times 1 = 1$

- $\lim\limits_{x \to 0} \dfrac{1 - \cos 4x}{x^2} = \lim\limits_{x \to 0} \dfrac{(1 - \cos 4x) \times 16}{4^2 \times x^2}$
  $= \lim\limits_{x \to 0} \dfrac{(1 - \cos 4x) \times 16}{(4x)^2} = \dfrac{1}{2} \times 16$
  $= \boxed{8}$

- $\lim\limits_{x \to 0} \dfrac{\cos 2x - 1}{x^2} = \lim\limits_{x \to 0} \dfrac{-(1 - \cos 2x)}{x^2}$
  $= \lim\limits_{x \to 0} \dfrac{-(1 - \cos 2x)}{2^2 \times x^2}$
  $= \lim\limits_{x \to 0} \dfrac{-(1 - \cos 2x)}{(2x)^2} = \boxed{-\dfrac{1}{2}}$

$$\lim_{x \to 0} \frac{\sin(x^2+3x)}{x} = ??$$

Règle N° 3 :

$$\begin{cases} \lim_{x \to 0} \frac{\sin(ax)}{\tan(bx)} = \frac{a}{b} \\ \lim_{x \to 0} \frac{\sin(ax)}{\sin(bx)} = \frac{a}{b} \\ \lim_{x \to 0} \frac{\tan(bx)}{\sin ax} = b/a \end{cases}$$

FIN !!

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